On some multiple zeta-star values of one-two-three indices

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Abstract

In this paper, we present some identities for multiple zeta-star values with indices obtained by inserting 3 or 1 into the string $2, \ldots, 2$. Our identities give analogues of Zagier's evaluation of $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$ and examples of a kind of duality of multiple zeta-star values. Moreover, their generalizations give partial solutions of conjectures proposed by Imatomi, Tanaka, Wakabayashi and the first author.

1 Introduction

Multiple zeta values and multiple zeta-star values (MZVs and MZSVs for short) are defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$
$$\zeta^\star(k_1, k_2, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \ldots, k_n$ are positive integers with $k_1 \geq 2$. The weight and depth of the above series are by definition the integers $k = k_1 + \cdots + k_n$ and n, respectively.

We are interested in $\mathbb{Q}$-linear relations among these real values. This topic has been studied by many mathematicians and physicists. In this paper, we prove the following new $\mathbb{Q}$-linear relations among MZSVs, which are conjectured by M. Kaneko in his unpublished work:

Theorem 1.1. *For any positive integers n and m, we have*

$$\zeta^\star(\{2\}^m, 1) \cdot \zeta^\star(\{2\}^n, 1) = \zeta^\star(\{2\}^m, 1, \{2\}^n, 1) + \zeta^\star(\{2\}^n, 1, \{2\}^m, 1), \tag{1}$$
$$\zeta^\star(\{2\}^m, 1) \cdot \zeta^\star(\{2\}^n) = \zeta^\star(\{2\}^m, 1, \{2\}^n) + \zeta^\star(\{2\}^{n-1}, 3, \{2\}^m), \tag{2}$$
$$\zeta^\star(\{2\}^m) \cdot \zeta^\star(\{2\}^n) = \zeta^\star(\{2\}^{m-1}, 3, \{2\}^{n-1}, 1) + \zeta^\star(\{2\}^{n-1}, 3, \{2\}^{m-1}, 1), \tag{3}$$

where $\{2\}^n$ stands for the n-tuple of 2.

The identity (2) will be shown in Section 2, and (1) and (3) in Section 3. In this introduction, we make some remarks on these formulas.

First, we note that the identity (1) is already shown by Ohno and Zudilin ([8]) using their 'two-one formula'

$$\zeta^\star(\{2\}^m, 1, \{2\}^n, 1) = 4\zeta^\star(2m+1, 2n+1) - 2\zeta(2m+2n+2)$$

of depth 2. Our proof of (1) is, however, fairly simpler than theirs and also works for proving (3) almost identically. Moreover, the same method leads to the following generalizations of (1) and (3).

Theorem 1.2. *Let $n > 0$ be an integer.*

(i) *For any non-negative integers $j_1, j_2, \ldots, j_n$ such that $j_1, j_n \geq 1$, we have*

$$\sum_{k=0}^{n} (-1)^k \zeta^\star(\{2\}^{j_1}, 1, \ldots, \{2\}^{j_k}, 1) \cdot \zeta^\star(\{2\}^{j_n}, 1, \ldots, \{2\}^{j_{k+1}}, 1) = 0. \tag{4}$$

(ii) *For any non-negative integers $j_1, j_2, \ldots, j_{2n}$, we have*

$$\begin{aligned}&\sum_{k=0}^{n} \zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1, \ldots, 3, \{2\}^{j_{2k}}, 1) \cdot \zeta^\star(\{2\}^{j_{2n}}, 3, \{2\}^{j_{2n-1}}, 1, \ldots, 3, \{2\}^{j_{2k+1}}, 1)\\ &= \sum_{k=1}^{n} \zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1, \ldots, 1, \{2\}^{j_{2k-1}+1}) \cdot \zeta^\star(\{2\}^{j_{2n}}, 3, \{2\}^{j_{2n-1}}, 1, \ldots, 1, \{2\}^{j_{2k+1}+1}).\end{aligned} \tag{5}$$

In fact, we prove even more general formula than Theorem 1.2 (see Theorem 3.1).

The identity (5) has an application to the following conjecture:

Conjecture 1.3 ([5, Conjecture 4.5])**.** (*A*) *Let n be a positive integer, and $j_0, j_1, \ldots, j_{2n-1}$ non-negative integers. Put $m = j_0 + j_1 + \cdots + j_{2n-1}$. Then we have*

$$\sum_{\sigma \in \mathfrak{S}_{2n}} \zeta^\star(\{2\}^{j_{\sigma(0)}}, 3, \{2\}^{j_{\sigma(1)}}, 1, \{2\}^{j_{\sigma(2)}}, \ldots, 3, \{2\}^{j_{\sigma(2n-1)}}, 1) \overset{?}{\in} \mathbb{Q} \cdot \pi^{2m+4n}.$$

(*B*) *Let $n, j_0, j_1, \ldots, j_{2n}$ be non-negative integers. Put $m = j_0 + j_1 + \cdots + j_{2n}$. Then we have*

$$\sum_{\sigma \in \mathfrak{S}_{2n+1}} \zeta^\star(\{2\}^{j_{\sigma(0)}}, 3, \{2\}^{j_{\sigma(1)}}, 1, \{2\}^{j_{\sigma(2)}}, \ldots, 3, \{2\}^{j_{\sigma(2n-1)}}, 1, \{2\}^{j_{\sigma(2n)}+1}) \overset{?}{\in} \mathbb{Q} \cdot \pi^{2m+4n+2}.$$

We denote by (A_n) (resp. (B_n)) the statement (A) (resp. (B)) in Conjecture 1.3 for a specific value of n. For example, (B_0) means that $\zeta^\star(\{2\}^{j+1}) \in \mathbb{Q} \cdot \pi^{2j+2}$ for $j \geq 0$, which is already known (see [12]). On the other hand, (A_1) is a consequence of (B_0) and the identity (3) of Theorem 1.1. This implication $(B_0) \implies (A_1)$ is generalized as follows:

Theorem 1.4. *Let n be a positive integer. If the statements (A_l) and (B_l) hold for all $l < n$, then (A_n) is also true.*

Next, we explain two topics related with the identity (2). The first is on the 'duality' for MZSVs. The first example of such phenomena was found by Kaneko-Ohno [6], who proved that

$$(-1)^{n+1}\zeta^\star(m+1,\{1\}^n)-(-1)^{m+1}\zeta^\star(n+1,\{1\}^m)\in\mathbb{Q}[\zeta(2),\zeta(3),\zeta(5),\ldots]. \tag{6}$$

This is regarded as an analogue of the duality of MZVs

$$\zeta(m+1,\{1\}^{n-1})-\zeta(n+1,\{1\}^{m-1})=0.$$

They also formulated a conjecture which generalizes (6), and recently Li [7] and Yamazaki [10] proved this conjecture using generalized hypergeometric functions.

On the other hand, since $\zeta^\star(\{2\}^m,1)=2\zeta(2m+1)$ and $\zeta^\star(\{2\}^n)=2(1-2^{1-2n})\zeta(2n)$, our formula (2) implies the following:

Corollary 1.5. *For integers $m,n\geq 1$, we have*

$$(-1)^{m+n+1}\zeta^\star(\{2\}^m,1,\{2\}^n)-(-1)^{m+n}\zeta^\star(\{2\}^{n-1},3,\{2\}^m)\in\mathbb{Q}[\zeta(2),\zeta(3),\zeta(5),\ldots]. \tag{7}$$

This might be regarded as an analogue of the duality

$$\zeta(\{2\}^m,1\{2\}^n)-\zeta(\{2\}^n,3,\{2\}^{m-1})=0. \tag{8}$$

We remark that (7) is not contained in the Kaneko-Ohno conjecture. This suggests that the 'duality' for MZSVs might be open to further extension, but we will not develop this point in the present paper.

The second topic is inspired by Zagier's work [11] on an evaluation of $\zeta(\{2\}^m,3,\{2\}^n)$ which plays an important role in Brown's partial settlement of Hoffman's basis conjecture for MZVs ([1], see also [2]). In fact, Zagier also proved an analogous evaluation of MZSV:

$$\begin{aligned}&\zeta^\star(\{2\}^m,3,\{2\}^n)\\&\quad=-2\sum_{r=1}^{m+n+1}\left(\binom{2r}{2n}-\delta_{r,n}-(1-4^{-r})\binom{2r}{2m+1}\right)\zeta(2r+1)\,\zeta^\star(\{2\}^{m+n+1-r}).\end{aligned} \tag{9}$$

Note that, since the duality (8) holds, the formula for $\zeta(\{2\}^m,3,\{2\}^n)$ may also be regarded as an evaluation of $\zeta(\{2\}^{m+1},1,\{2\}^n)$. On the other hand, an evaluation for $\zeta^\star(\{2\}^{m+1},1,\{2\}^n)$ can be obtained by combining (9) with our result (2):

Theorem 1.6. *For any non-negative integers m,n, we have*

$$\begin{aligned}&\zeta^\star(\{2\}^{m+1},1,\{2\}^n)\\&\quad=2\sum_{r=1}^{m+n+1}\left(\binom{2r}{2m+2}-(1-4^{-r})\binom{2r}{2n-1}\right)\zeta(2r+1)\,\zeta^\star(\{2\}^{m+n+1-r}).\end{aligned} \tag{10}$$

We also remark that Theorem 1.6 implies the following corollary, in the same way that (9) implies the corresponding result [11, Theorem 2]:

Corollary 1.7. *For each odd integer $k \geq 3$, the $\mathbb{Q}$-vector space spanned by $\zeta^\star(\{2\}^{m+1}, 1, \{2\}^n)$ with $2m + 2n + 3 = k$ and $m, n \geq 0$ is equal to the $\mathbb{Q}$-vector space spanned by $\pi^{2r}\zeta(k - 2r)$ ($r = 0, 1, \ldots, (k-3)/2$).*

2 On the identity (2)

In this section, we prove the identity (2) in Theorem 1.1.

For an integer $p \geq 0$, we denote by $\zeta_p(k_1, \ldots, k_n)$ (resp. $\zeta_p^\star(k_1, \ldots, k_n)$) the finite sum obtained by truncating the series for $\zeta(k_1, \ldots, k_n)$ (resp. $\zeta^\star(k_1, \ldots, k_n)$):

$$\zeta_p(k_1, k_2, \ldots, k_n) = \sum_{p \geq p_1 > p_2 > \cdots > p_n > 0} \frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}}$$

$$\left(\text{resp. } \zeta_p^\star(k_1, k_2, \ldots, k_n) = \sum_{p \geq p_1 \geq p_2 \geq \cdots \geq p_n \geq 1} \frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}}\right).$$

The empty sum is interpreted as 0, and for the unique index $\varnothing$ of depth 0, we put $\zeta_p(\varnothing) = \zeta_p^\star(\varnothing) = 1$ even if $p = 0$.

Let a, b and c be positive integers. Our strategy for proving the identity (2) is to study the four generating functions

$$F_p(x, y) = \sum_{m,n \geq 0} \zeta_p(\{a\}^m, b, \{c\}^n) x^m y^n, \quad G_p(y) = \sum_{n \geq 0} \zeta_p(\{c\}^n) y^n,$$
$$F_p^\star(x, y) = \sum_{m,n \geq 0} \zeta_p^\star(\{c\}^m, b, \{a\}^n) x^m y^n, \quad G_p^\star(y) = \sum_{n \geq 0} \zeta_p^\star(\{a\}^n) y^n.$$

Note that $F_0(x, y) = F_0^\star(x, y) = 0$ and $G_0(y) = G_0^\star(y) = 1$.

Lemma 2.1. *For any integer $p \geq 0$, we have*

$$\begin{pmatrix} F_p(x, y) \\ G_p(y) \end{pmatrix} = T_p T_{p-1} \cdots T_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{11}$$

$$\begin{pmatrix} F_p^\star(x, y) \\ G_p^\star(y) \end{pmatrix} = U_p U_{p-1} \cdots U_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{12}$$

where

$$T_q = T_q(x, y) = \begin{pmatrix} 1 + \frac{x}{q^a} & \frac{1}{q^b} \\ 0 & 1 + \frac{y}{q^c} \end{pmatrix},$$

$$U_q = U_q(x, y) = \left(1 - \frac{x}{q^c}\right)^{-1} \left(1 - \frac{y}{q^a}\right)^{-1} \begin{pmatrix} 1 - \frac{y}{q^a} & \frac{1}{q^b} \\ 0 & 1 - \frac{x}{q^c} \end{pmatrix}.$$

Proof. The case $p=0$ is obvious. For $p>0$, by definition, we have

$$\begin{aligned}
F_p(x,y) &= \sum_{m,n\geq 0}\sum_{p\geq p_1>\cdots>p_{m+n+1}>0}\frac{x^m y^n}{p_1^a\cdots p_m^a p_{m+1}^b p_{m+2}^c\cdots p_{m+n+1}^c}\\
&= F_{p-1}(x,y)+\sum_{m,n\geq 0}\sum_{p=p_1>\cdots>p_{m+n+1}>0}\frac{x^m y^n}{p_1^a\cdots p_m^a p_{m+1}^b p_{m+2}^c\cdots p_{m+n+1}^c}\\
&= F_{p-1}(x,y)+\sum_{n\geq 0}\sum_{p=p_1>\cdots>p_{n+1}>0}\frac{y^n}{p_1^b p_2^c\cdots p_{n+1}^c}\\
&\quad +\sum_{m>0,n\geq 0}\sum_{p=p_1>\cdots>p_{m+n+1}>0}\frac{x^m y^n}{p_1^a\cdots p_m^a p_{m+1}^b p_{m+2}^c\cdots p_{m+n+1}^c}\\
&= F_{p-1}(x,y)+\frac{1}{p^b}G_{p-1}(y)+\frac{x}{p^a}F_{p-1}(x,y).
\end{aligned}$$

In a similar but simpler way, we also obtain $G_p(y)=G_{p-1}(y)+\frac{y}{p^c}G_{p-1}(y)$. Hence we have

$$\begin{pmatrix}F_p(x,y)\\ G_p(y)\end{pmatrix}=\begin{pmatrix}1+\frac{x}{p^a} & \frac{1}{p^b}\\ 0 & 1+\frac{y}{p^c}\end{pmatrix}\begin{pmatrix}F_{p-1}(x,y)\\ G_{p-1}(y)\end{pmatrix},$$

and the identity (11) by induction.

The case (12) can be shown in a similar way. In fact, one has

$$\begin{aligned}
F_p^\star(x,y) &= F_{p-1}^\star(x,y)+\frac{1}{p^b}G_p^\star(y)+\frac{x}{p^c}F_p^\star(x,y),\\
G_p^\star(y) &= G_{p-1}^\star(y)+\frac{y}{p^a}G_p^\star(y),
\end{aligned}$$

that is

$$\begin{pmatrix}1-\frac{x}{p^c} & -\frac{1}{p^b}\\ 0 & 1-\frac{y}{p^a}\end{pmatrix}\begin{pmatrix}F_p^\star(x,y)\\ G_p^\star(y)\end{pmatrix}=\begin{pmatrix}F_{p-1}^\star(x,y)\\ G_{p-1}^\star(y)\end{pmatrix}.$$

It is easy to see that the inverse of the 2×2 matrix in the left hand side is equal to U_p. $\square$

Proposition 2.2. *For any non-negative integers p, m and n, we have*

$$\zeta_p^\star(\{c\}^m,b,\{a\}^n)=\sum_{k=0}^{m}\sum_{l=0}^{n}(-1)^{k+l}\zeta_p(\{a\}^l,b,\{c\}^k)\cdot\zeta_p^\star(\{c\}^{m-k})\cdot\zeta_p^\star(\{a\}^{n-l}). \qquad (13)$$

Proof. Lemma 2.1 implies that

$$F_p^\star(x,y)=F_p(-y,-x)\prod_{q=1}^{p}\left(1-\frac{x}{q^c}\right)^{-1}\prod_{q=1}^{p}\left(1-\frac{y}{q^a}\right)^{-1}. \qquad (14)$$

Here we have

$$\prod_{q=1}^{p}\left(1-\frac{x}{q^c}\right)^{-1}=\sum_{m\geq 0}\zeta_p^\star(\{c\}^m)x^m,\quad \prod_{q=1}^{p}\left(1-\frac{y}{q^a}\right)^{-1}=\sum_{n\geq 0}\zeta_p^\star(\{a\}^n)y^n.$$

By comparing the coefficients of (14), we obtain (13). $\square$

Remark 2.3. Here we make a remark on an algebraic interpretation of the identity (13).

First we recall the setup of harmonic algebra (see [3] for details). Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial algebra in two indeterminates x, y, and $\mathfrak{H}^1$ its subalgebra $\mathbb{Q} + \mathfrak{H}y$. For an integer $p \geq 0$, we define the $\mathbb{Q}$-linear maps $Z_p\colon \mathfrak{H}^1 \longrightarrow \mathbb{Q}$ and $Z_p^\star\colon \mathfrak{H}^1 \longrightarrow \mathbb{Q}$ by

$$Z_p(1) = 1 \text{ and } \ Z_p(z_{k_1}\cdots z_{k_n}) = \zeta_p(k_1,\ldots,k_n),$$
$$Z_p^\star(1) = 1 \text{ and } \ Z_p^\star(z_{k_1}\cdots z_{k_n}) = \zeta_p^\star(k_1,\ldots,k_n),$$

where $z_k = x^{k-1}y$ $(k = 1, 2, \ldots)$. Let γ be the algebra automorphism on $\mathfrak{H}$ characterized by $\gamma(x) = x$ and $\gamma(y) = x + y$, and define the $\mathbb{Q}$-linear transformation $d\colon \mathfrak{H}^1 \longrightarrow \mathfrak{H}^1$ by

$$d(1) = 1 \text{ and } d(wy) = \gamma(w)y$$

for any word $w \in \mathfrak{H}$. Then one has $Z_p \circ d = Z_p^\star$. Moreover, we define the harmonic product $*$, a $\mathbb{Q}$-bilinear product on $\mathfrak{H}^1$, inductively by

$$1 * w = w * 1 = w,$$
$$z_k w * z_l w' = z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w'),$$

where $k, l \geq 1$ and $w, w' \in \mathfrak{H}^1$. This product $*$ makes $\mathfrak{H}^1$ a commutative $\mathbb{Q}$-algebra, and $Z_p\colon \mathfrak{H}^1 \longrightarrow \mathbb{Q}$ an algebra homomorphism for any $p \geq 0$.

Now the algebraic counterpart to the identity (13) is the following:

Proposition 2.4. *For any $m, n \geq 0$, we have*

$$d(z_c^m z_b z_a^n) = \sum_{k=0}^{m}\sum_{l=0}^{n}(-1)^{k+l} z_a^l z_b z_c^k * d(z_c^{m-k}) * d(z_a^{n-l}). \tag{15}$$

In fact, this proposition is equivalent to the identity (13), since the map from $\mathfrak{H}^1$ to $\mathbb{Q}^{\mathbb{N}}$ that sends w to $\{Z_p(w)\}_p$ is injective (see [9, §3] for details). Alternatively, one can also prove (15) directly (this last remark was communicated by Shingo Saito).

Now we return to the proof of the identity (2) of Theorem 1.1.

Proof of (2). By Proposition 2.2, we have for any integers $m, n > 0$

$$\begin{aligned}
\zeta_p^\star(\{2\}^m, 1, \{2\}^n) &= \sum_{k=0}^{m}\sum_{l=0}^{n}(-1)^{k+l}\zeta_p(\{2\}^l, 1, \{2\}^k)\,\zeta_p^\star(\{2\}^{m-k})\,\zeta_p^\star(\{2\}^{n-l}),\\
\zeta_p^\star(\{2\}^{n-1}, 3, \{2\}^m) &= \sum_{k=0}^{m}\sum_{l=0}^{n-1}(-1)^{k+l}\zeta_p(\{2\}^k, 3, \{2\}^l)\,\zeta_p^\star(\{2\}^{m-k})\,\zeta_p^\star(\{2\}^{n-1-l})\\
&= \sum_{k=0}^{m}\sum_{l=1}^{n}(-1)^{k+l+1}\zeta_p(\{2\}^k, 3, \{2\}^{l-1})\,\zeta_p^\star(\{2\}^{m-k})\,\zeta_p^\star(\{2\}^{n-l}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\zeta_p^\star(\{2\}^m,1,\{2\}^n)+\zeta_p^\star(\{2\}^{n-1},3,\{2\}^m)\\
&\ =\sum_{k=0}^{m}(-1)^k\zeta_p(1,\{2\}^k)\,\zeta_p^\star(\{2\}^{m-k})\,\zeta_p^\star(\{2\}^n)\\
&\quad\ +\sum_{k=0}^{m}\sum_{l=1}^{n}(-1)^{k+l}\Big\{\zeta_p(\{2\}^l,1,\{2\}^k)-\zeta_p(\{2\}^k,3,\{2\}^{l-1})\Big\}\zeta_p^\star(\{2\}^{m-k})\,\zeta_p^\star(\{2\}^{n-l}).
\end{aligned}\tag{16}$$

By the duality (8), the second term in the right hand side of (16) vanishes when $p\to\infty$. Finally, using the identity

$$\zeta_p^\star(\{2\}^m,1)=\sum_{k=0}^{m}(-1)^k\zeta_p(1,\{2\}^k)\,\zeta_p^\star(\{2\}^{m-k}),$$

which is a special case of Proposition 2.2, we prove the identity (2) by letting $p\to\infty$. $\square$

Remark 2.5. Although both sides of the identity (2) diverge when $m=0$, the above proof shows that for any non-negative integer p we have

$$\begin{aligned}
&\zeta_p^\star(1,\{2\}^n)+\zeta_p^\star(\{2\}^{n-1},3)\\
&=\zeta_p^\star(1)\zeta_p^\star(\{2\}^n)+\sum_{l=1}^{n}(-1)^l\Big\{\zeta_p(\{2\}^l,1)-\zeta_p(3,\{2\}^{l-1})\Big\}\zeta_p^\star(\{2\}^{n-l}).
\end{aligned}\tag{17}$$

On the other hand, the harmonic product relation implies that

$$\zeta_p^\star(1)\zeta_p^\star(\{2\}^n)=\sum_{l=0}^{n}\zeta_p^\star(\{2\}^l,1,\{2\}^{n-l})-\sum_{l=0}^{n-1}\zeta_p^\star(\{2\}^l,3,\{2\}^{n-1-l}).$$

Substituting it into (17), we obtain

$$\begin{aligned}
\zeta_p^\star(\{2\}^{n-1},3)&=\sum_{l=1}^{n}\zeta_p^\star(\{2\}^l,1,\{2\}^{n-l})-\sum_{l=0}^{n-1}\zeta_p^\star(\{2\}^l,3,\{2\}^{n-1-l})\\
&\quad+\sum_{l=1}^{n}(-1)^l\Big\{\zeta_p(\{2\}^l,1)-\zeta_p(3,\{2\}^{l-1})\Big\}\zeta_p^\star(\{2\}^{n-l}).
\end{aligned}$$

By letting $p\to\infty$, we get the following:

Proposition 2.6. *For any positive integer n, we have*

$$\zeta^\star(\{2\}^{n-1},3)=\sum_{l=1}^{n}\zeta^\star(\{2\}^l,1,\{2\}^{n-l})-\sum_{l=0}^{n-1}\zeta^\star(\{2\}^l,3,\{2\}^{n-1-l}).$$

This proposition was shown by Ihara, Kajikawa, Ohno and Okuda [4, eq. (12)] using the derivation relation for MZSVs.

3 On the identities (1) and (3)

In this section, we prove a formula which includes the identities (1) and (3) as special cases.

First we introduce some notation. For an integer $n \geq 1$, consider two vectors $\vec{\jmath} = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$ and $\vec{e} = (e_1, \dots, e_{n-1}) \in \{1,3\}^{n-1}$ (here $\{1,3\}^{n-1}$ does not mean the sequence $(1,3,\dots,1,3)$, but the $(n-1)$-th power of the set $\{1,3\}$). We set

$$\overline{\mathfrak{Z}}_n(\vec{\jmath}, \vec{e}) := \zeta^\star\big(\{2\}^{j_1}, e_1, \{2\}^{j_2}, e_2, \dots, e_{n-1}, \{2\}^{j_n}\big). \tag{18}$$

For $n = 0$, we simply put $\overline{\mathfrak{Z}}_0 = 1$. Note that the right hand side of (18) diverges if and only if $n \geq 2$, $j_1 = 0$ and $e_1 = 1$. If this is not the case, we say that $(\vec{\jmath}, \vec{e})$ is an admissible pair.

For $\vec{\jmath} = (j_1, \dots, j_n)$, we define the following operations:

$$\begin{aligned} \vec{\jmath}_+ &:= (j_1, \dots, j_n, 0), & \vec{\jmath}^{\,+} &:= (j_1, \dots, j_{n-1}, j_n + 1), \\ \vec{\jmath}' &:= (j_n, \dots, j_1), & \vec{\jmath}|_k &:= (j_1, \dots, j_k) \quad (k = 0, \dots, n). \end{aligned}$$

For example, we have $(\vec{\jmath}'|_{n-k})_+ = (j_n, \dots, j_{k+1}, 0)$. We also apply similar operations to $\vec{e}$ in obvious manners.

Our main result in this section is the following formula:

Theorem 3.1. *Let n be a positive integer, $\vec{\jmath} = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$ and $\vec{e} = (e_1, \dots, e_{n-1}) \in \{1,3\}^{n-1}$. Assume that both $(\vec{\jmath}, \vec{e})$ and $(\vec{\jmath}', \vec{e}')$ are admissible pairs. Put $e_0 = e_n = 1$, and*

$$X(k) := \begin{cases} \overline{\mathfrak{Z}}_{k+1}((\vec{\jmath}|_k)_+, \vec{e}|_k) \cdot \overline{\mathfrak{Z}}_{n-k+1}((\vec{\jmath}'|_{n-k})_+, \vec{e}'|_{n-k}) & (\text{if } e_k = 1), \\ \overline{\mathfrak{Z}}_k((\vec{\jmath}|_k)^+, \vec{e}|_{k-1}) \cdot \overline{\mathfrak{Z}}_{n-k}((\vec{\jmath}'|_{n-k})^+, \vec{e}'|_{n-k-1}) & (\text{if } e_k = 3) \end{cases}$$

for $k = 0, \dots, n$ (here $\vec{e}|_n$ and $\vec{e}'|_n$ stand for $(e_1, \dots, e_n)$ and $(e_{n-1}, \dots, e_0)$, respectively). Then we have

$$\sum_{k=0}^{n} (-1)^k X(k) = 0.$$

Let us examine some examples. First, we set $\vec{e} = (1, 1, \dots, 1)$. Then we have

$$\begin{aligned} X(k) &= \overline{\mathfrak{Z}}_{k+1}\big((j_1, \dots, j_k, 0), (1, \dots, 1)\big) \cdot \overline{\mathfrak{Z}}_{n-k+1}\big((j_n, \dots, j_{k+1}, 0), (1, \dots, 1)\big) \\ &= \zeta^\star(\{2\}^{j_1}, 1, \dots, \{2\}^{j_k}, 1) \cdot \zeta^\star(\{2\}^{j_n}, 1, \dots, \{2\}^{j_{k+1}}, 1). \end{aligned}$$

Hence Theorem 3.1 implies the identity (4) in Theorem 1.2 in this case. Similarly, for $\vec{e} = (3, 1, 3, \dots, 1, 3) \in \{1,3\}^{2n-1}$, we have

$$\begin{aligned} X(k) = &\zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1, \dots, 3, \{2\}^{j_k}, 1) \\ &\times \zeta^\star(\{2\}^{j_{2n}}, 3, \{2\}^{j_{2n-1}}, 1, \dots, 3, \{2\}^{j_{k+1}}, 1) \end{aligned}$$

if k is even, and

$$\begin{aligned} X(k) = &\zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1, \dots, 3, \{2\}^{j_{k-1}}, 1, \{2\}^{j_k+1}) \\ &\times \zeta^\star(\{2\}^{j_{2n}}, 3, \{2\}^{j_{2n-1}}, 1, \dots, 3, \{2\}^{j_{k+2}}, 1, \{2\}^{j_{k+1}+1}) \end{aligned}$$

if k is odd. Therefore, we obtain the identity (5) in Theorem 1.2.

We may also apply Theorem 3.1 to other cases. For example, by setting $\vec{e} = (3,3)$ or $\vec{e} = (3,1)$, one obtains

$$\begin{aligned}&\zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 3, \{2\}^{j_3}, 1) + \zeta^\star(\{2\}^{j_1+1}) \cdot \zeta^\star(\{2\}^{j_3}, 3, \{2\}^{j_2+1})\\ &\quad = \zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2+1}) \cdot \zeta^\star(\{2\}^{j_3+1}) + \zeta^\star(\{2\}^{j_3}, 3, \{2\}^{j_2}, 3, \{2\}^{j_1}, 1)\end{aligned}$$

or

$$\begin{aligned}&\zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1, \{2\}^{j_3}, 1) + \zeta^\star(\{2\}^{j_1+1}) \cdot \zeta^\star(\{2\}^{j_3}, 1, \{2\}^{j_2+1})\\ &\quad = \zeta^\star(\{2\}^{j_1}, 3, \{2\}^{j_2}, 1) \cdot \zeta^\star(\{2\}^{j_3}, 1) + \zeta^\star(\{2\}^{j_3}, 1, \{2\}^{j_2}, 3, \{2\}^{j_1}, 1),\end{aligned}$$

respectively.

Now we proceed to the proof of Theorem 3.1. For $\infty \geq A \geq B \geq 1$, we put

$$C_{-1}(A,B) = \delta_{A,B} A^2, \qquad C_0(A,B) = 1,$$
$$C_j(A,B) = \sum_{A \geq a_1 \geq \cdots \geq a_j \geq B} \frac{1}{a_1^2 \cdots a_j^2} \qquad (j = 1,2,3,\ldots).$$

Then we have

$$C_j(A,B) = \sum_{p=B}^{A} \frac{1}{p^2} C_{j-1}(p,B) = \sum_{p=B}^{A} C_{j-1}(A,p) \frac{1}{p^2} \tag{19}$$

for any $j \geq 0$.

Lemma 3.2. *For any $j \geq -1$ and $1 \leq p, q \leq \infty$, we have*

$$\sum_{p_0=1}^{p} C_j(p,p_0) \frac{q}{p_0(p_0+q)} = \sum_{q_0=1}^{q} C_j(q,q_0) \frac{p}{q_0(q_0+p)}. \tag{20}$$

Here, $\frac{p}{q_0(q_0+p)}$ means $\frac{1}{q_0}$ if $p = \infty$, and so on.

Proof. We use induction on j. When $j = -1$, both sides are equal to $\frac{pq}{p+q}$. Thus we assume $j \geq 0$. By (19), we can rewrite the left hand side of (20) as

$$\begin{aligned}\sum_{p_0=1}^{p} C_j(p,p_0) \frac{q}{p_0(p_0+q)} &= \sum_{p_0=1}^{p} \sum_{p_1=p_0}^{p} C_{j-1}(p,p_1) \frac{1}{p_1^2} \left(\frac{1}{p_0} - \frac{1}{p_0+q} \right)\\ &= \sum_{p_1=1}^{p} C_{j-1}(p,p_1) \frac{1}{p_1^2} \sum_{p_0=1}^{p_1} \left(\frac{1}{p_0} - \frac{1}{p_0+q} \right).\end{aligned}$$

By using the identity

$$\begin{aligned}\sum_{p_0=1}^{p_1} \left(\frac{1}{p_0} - \frac{1}{p_0+q} \right) &= \sum_{p_0=1}^{\infty} \left\{ \left(\frac{1}{p_0} - \frac{1}{p_0+q} \right) - \left(\frac{1}{p_0+p_1} - \frac{1}{p_0+p_1+q} \right) \right\}\\ &= \sum_{p_0=1}^{q} \left(\frac{1}{p_0} - \frac{1}{p_0+p_1} \right) = \sum_{q_1=1}^{q} \frac{p_1}{q_1(q_1+p_1)},\end{aligned}$$

we get

$$\sum_{p_1=1}^{p} C_{j-1}(p,p_1)\frac{1}{p_1^2}\sum_{p_0=1}^{p_1}\left(\frac{1}{p_0}-\frac{1}{p_0+q}\right) = \sum_{p_1=1}^{p} C_{j-1}(p,p_1)\frac{1}{p_1^2}\sum_{q_1=1}^{q}\frac{p_1}{q_1(q_1+p_1)}$$
$$= \sum_{q_1=1}^{q}\frac{1}{q_1^2}\sum_{p_1=1}^{p} C_{j-1}(p,p_1)\frac{q_1}{p_1(p_1+q_1)}.$$

Finally, using the induction hypothesis and (19) again, we obtain

$$\sum_{q_1=1}^{q}\frac{1}{q_1^2}\sum_{p_1=1}^{p} C_{j-1}(p,p_1)\frac{q_1}{p_1(p_1+q_1)} = \sum_{q_1=1}^{q}\frac{1}{q_1^2}\sum_{q_0=1}^{q_1} C_{j-1}(q_1,q_0)\frac{p}{q_0(q_0+p)}$$
$$= \sum_{q_0=1}^{q} C_j(q,q_0)\frac{p}{q_0(q_0+p)}.$$

Thus we have shown (20). □

Proof of Theorem 3.1. For $k=0,\dots,n$, put

$$E(k) = \sum_{\substack{\infty=p_0\ge\cdots\ge p_k\ge1\\ \infty=q_{n+1}\ge q_n\ge\cdots\ge q_{k+1}\ge1}} \prod_{\alpha=1}^{k} C_{j_\alpha}(p_{\alpha-1},p_\alpha)p_\alpha^{-e_\alpha}\prod_{\beta=k+1}^{n} C_{j_\beta}(q_{\beta+1},q_\beta)q_\beta^{-e_{\beta-1}}\cdot\frac{p_k^{e_k-1}q_{k+1}}{p_k+q_{k+1}},$$

$$F(k) = \sum_{\substack{\infty=p_0\ge\cdots\ge p_k\ge1\\ \infty=q_{n+1}\ge q_n\ge\cdots\ge q_{k+1}\ge1}} \prod_{\alpha=1}^{k} C_{j_\alpha}(p_{\alpha-1},p_\alpha)p_\alpha^{-e_\alpha}\prod_{\beta=k+1}^{n} C_{j_\beta}(q_{\beta+1},q_\beta)q_\beta^{-e_{\beta-1}}\cdot\frac{p_kq_{k+1}^{e_k-1}}{p_k+q_{k+1}}.$$

In particular, we have $E(0)=0$ (resp. $F(n)=0$) because of the factor $\frac{q_1}{\infty+q_1}$ (resp. $\frac{p_n}{p_n+\infty}$) (recall that we set $e_0=e_n=1$). For general k, one can verify that $E(k)+F(k)=X(k)$ by using

$$\frac{p_k^{e_k-1}q_{k+1}}{p_k+q_{k+1}}+\frac{p_kq_{k+1}^{e_k-1}}{p_k+q_{k+1}} = \begin{cases}1 & (\text{if } e_k=1),\\ p_kq_{k+1} & (\text{if } e_k=3).\end{cases}$$

Moreover, when $1\le k\le n$, one has

$$\sum_{p_k=1}^{p_{k-1}} C_{j_k}(p_{k-1},p_k)p_k^{-e_k}\frac{p_k^{e_k-1}q_{k+1}}{p_k+q_{k+1}} = \sum_{q_k=1}^{q_{k+1}} C_{j_k}(q_{k+1},q_k)q_k^{-e_{k-1}}\frac{p_{k-1}q_k^{e_{k-1}-1}}{p_{k-1}+q_k}$$

by Lemma 3.2, hence $E(k)=F(k-1)$. Therefore,

$$\sum_{k=0}^{n}(-1)^kX(k) = \sum_{k=0}^{n}(-1)^k\big(E(k)+F(k)\big) = \sum_{k=1}^{n}(-1)^kF(k-1)+\sum_{k=0}^{n-1}(-1)^kF(k)=0$$

as desired. □

Finally, we prove Theorem 1.4 in the introduction.

Proof of Theorem 1.4. Since we only use $\vec{e} = (3, 1, \dots, 3, 1)$ here, we omit it in this proof.

For $\vec{\jmath} = (j_0, \dots, j_n)$ and $\sigma \in \mathfrak{S}_{n+1}$, we put $\sigma(\vec{\jmath}) := (j_{\sigma(0)}, \dots, j_{\sigma(n)})$. Then the statement (A_n) is expressed as

$$\sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2n+1}\big(\sigma(\vec{\jmath})_+\big) \in \mathbb{Q} \cdot \pi^{2m+4n} \qquad \big(\forall \vec{\jmath} = (j_0, \dots, j_{2n-1}) \in \mathbb{Z}_{\geq 0}^{2n}\big). \tag{21}$$

Similarly, (B_n) says that

$$\sum_{\sigma \in \mathfrak{S}_{2n+1}} \overline{\mathfrak{Z}}_{2n+1}\big(\sigma(\vec{\jmath})^+\big) \in \mathbb{Q} \cdot \pi^{2m+4n+2} \qquad \big(\forall \vec{\jmath} = (j_0, \dots, j_{2n}) \in \mathbb{Z}_{\geq 0}^{2n+1}\big). \tag{22}$$

Now suppose that (A_l) and (B_l) hold for all $l < n$. For $\vec{\jmath} = (j_0, \dots, j_{2n-1})$, we rewrite the identity (5) as

$$\begin{aligned}\overline{\mathfrak{Z}}_{2n+1}(\vec{\jmath}_+) + \overline{\mathfrak{Z}}_{2n+1}(\vec{\jmath}'_+) = &\sum_{l=1}^{n} \overline{\mathfrak{Z}}_{2l-1}\big((\vec{\jmath}|_{2l-1})^+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\vec{\jmath}'|_{2n-2l+1})^+\big) \\ &- \sum_{l=1}^{n-1} \overline{\mathfrak{Z}}_{2l+1}\big((\vec{\jmath}|_{2l})_+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\vec{\jmath}'|_{2n-2l})_+\big).\end{aligned}$$

Summing up over $\sigma \in \mathfrak{S}_{2n}$, we obtain

$$\begin{aligned}2\sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2n+1}\big(\sigma(\vec{\jmath})_+\big) = &\sum_{l=1}^{n} \sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2l-1}\big((\sigma(\vec{\jmath})|_{2l-1})^+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\sigma(\vec{\jmath})'|_{2n-2l+1})^+\big) \\ &- \sum_{l=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2l+1}\big((\sigma(\vec{\jmath})|_{2l})_+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\sigma(\vec{\jmath})'|_{2n-2l})_+\big).\end{aligned} \tag{23}$$

Fix an integer l such that $1 \leq l \leq n$, and take a subset $S = \{s_0, \dots, s_{2(l-1)}\}$ of $\{0, \dots, 2n-1\}$ of cardinality $2l-1$. We also write $\{0, \dots, 2n-1\} \setminus S = \{t_0, \dots, t_{2(n-l)}\}$, and put $\vec{\jmath}_1 = (j_{s_0}, \dots, j_{s_{2(l-1)}})$, $\vec{\jmath}_2 = (j_{t_0}, \dots, j_{t_{2(n-l)}})$. Then we have

$$\begin{aligned}&\sum_{\substack{\sigma \in \mathfrak{S}_{2n} \\ \{\sigma(0), \dots, \sigma(2(l-1))\} = S}} \overline{\mathfrak{Z}}_{2l-1}\big((\sigma(\vec{\jmath})|_{2l-1})^+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\sigma(\vec{\jmath})'|_{2n-2l+1})^+\big) \\ &\qquad = \sum_{\tau_1 \in \mathfrak{S}_{2(l-1)+1}} \overline{\mathfrak{Z}}_{2(l-1)+1}\big(\tau_1(\vec{\jmath}_1)^+\big) \sum_{\tau_2 \in \mathfrak{S}_{2(n-l)+1}} \overline{\mathfrak{Z}}_{2(n-l)+1}\big(\tau_2(\vec{\jmath}_2)^+\big).\end{aligned} \tag{24}$$

If we set $m_1 = \sum_i j_{s_i}$ and $m_2 = \sum_i j_{t_i}$, the assumptions (B_{l-1}) and (B_{n-l}) implies that the right hand side of (24) belongs to $\mathbb{Q} \cdot \pi^{2m_1+4(l-1)+2} \cdot \pi^{2m_2+4(n-l)+2} = \mathbb{Q} \cdot \pi^{2m+4n}$. Therefore, by summing up over all l and S, we obtain that

$$\sum_{l=1}^{n} \sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2l-1}\big((\sigma(\vec{\jmath})|_{2l-1})^+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\sigma(\vec{\jmath})'|_{2n-2l+1})^+\big) \in \mathbb{Q} \cdot \pi^{2m+4n}.$$

Similarly, (A_l) for $l = 1, \ldots, n-1$ imply that

$$\sum_{l=1}^{n-1} \sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2l+1}\big((\sigma(\vec{\jmath})|_{2l})_+\big) \cdot \overline{\mathfrak{Z}}_{2n-2l+1}\big((\sigma(\vec{\jmath})'|_{2n-2l})_+\big) \in \mathbb{Q} \cdot \pi^{2m+4n}.$$

Thus, from (23), we conclude that

$$\sum_{\sigma \in \mathfrak{S}_{2n}} \overline{\mathfrak{Z}}_{2n+1}\big(\sigma(\vec{\jmath})_+\big) \in \mathbb{Q} \cdot \pi^{2m+4n}.$$

Now the proof of Theorem 1.4 is complete. □

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